J Glob Optim (2010) 47:421–435
DOI 10.1007/s10898-008-9366-y

Optimal control of a quasi-variational obstacle problem

Samir Adly · Maïtine Bergounioux · Mohamed Ait Mansour

Received: 4 September 2008 / Accepted: 3 October 2008 / Published online: 4 November 2008

Abstract We consider an optimal control where the state-control relation is given by a quasi-variational inequality, namely a generalized obstacle problem. We give an existence result for solutions to such a problem. The main tool is a stability result, based on the Mosco-convergence theory, that gives the weak closeness of the control-to-state operator. We end the paper with some examples.

Keywords Optimal control · Quasi-variational inequalities · Mosco convergence

1 Introduction

Optimal control of problems governed by PDE's have been extensively studied for many years. Then people investigated problems governed by variational inequalities (see [5] for example) from many points of view. Next challenge is the optimal control of problems whose state "equation" is a quasi-variational inequality (QVI). A first step has been done, considering problems where the control function is part of the variational inequality [7]. Now we are interested in the following

$$(\mathcal{P}) \qquad \min\{J(y, f),\ y \in \mathcal{T}(f),\ f \in U_{ad} \subset U\},$$

S. Adly (✉)
XLIM UMR-CNRS 6172, Université de Limoges, 87060 Limoges, France
e-mail: samir.adly@unilim.fr

M. Bergounioux
UMR 6628-MAPMO, Fédération Denis Poisson, Université d'Orléans, BP 6759,
45067 Orléans Cedex 2, France
e-mail: maitine.bergounioux@univ-orleans.fr

M. Ait Mansour
Faculté Poly-Disciplinaire,
Route Sidi Bouzid, BP 4162, 46000 Safi, Morocco
e-mail: maitmansour@hotmail.com

where $\mathcal{T}$ is a set-valued operator which associates to f, the set of elements y solution(s) to

$$\forall z \in K(y, f), \quad \langle \mathcal{A}(y, f), z - y \rangle \geq 0;$$

here K is a *set-valued application* from $X \times U$ to 2^X, X and U are Banach and Hilbert spaces respectively. Let us give an example: let Y be a Banach space and A a differential operator (linear or not), parabolic or elliptic from Y to the dual space Y', and Ξ an application from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. We identify Ξ and the associated Nemitsky operator. The differential equation that relates the control f to the state function y (i.e., the state "equation") is

$$\langle Ay, z - y \rangle_{Y,Y'} + \Xi(y, z, f) - \Xi(z, z, f) \geq (f, z - y) \quad \forall z \in Y,$$

where, for example

1. $\Xi(y, z, f) = \Xi(z)$ gives the classical variational inequalities;
2. $\Xi(y, z, f) = \Xi(f, z)$ gives (for example) obstacle problems (where the obstacle is the control) as in [7];

The full dependence of Ξ with respect to (y, z, f) leads to quasi-variational inequalities: this is the problem we are interested in.

To get existence results for solutions to problem $(\mathcal{P})$ we need continuity/stability properties for the state-control operator $\mathcal{T}$. So, we have to study precisely the quasi-variational inequalities from this point of view.

Let us mention that few people has been investigating optimal control problems for quasi-variational inequalities. Dietrich [10] has been considering problems where $\Xi(y, z, f)$ is the value at z of the indicatrix function of a set $K(y) = g(y) + C$ where C is constant and g is a $\mathcal{C}^1$ function, using a smooth dual gap function [9]. In our paper, we adopt an abstract point of view and give generic assumptions to get existence in a general context.

The paper is organized as follows. We first present the problem and recall classical tools and definitions. In Sect. 3, we give an existence result for solution to the quasi-variational inequality. Next section is devoted to stability results that allow to give a weak closeness property of the state-control operator $\mathcal{T}$. In last section we prove that the optimal control problem has at least one optimal solution and give many examples.

2 The basic quasi-variational inequality problem

In this section we present the QVI and recall some classical definitions. Then we give an existence result for this QVI.

2.1 Setting the QVI problem

Let us recall what a quasi-variational inequality is in an abstract setting.

Given a closed convex set D of a vector topological space X, a real-valued function $\varphi : D \times D \to \mathbb{R}$ and an extended real-valued function $\Sigma : D \times D \to \mathbb{R} \cup \{+\infty\}$, we introduce the following abstract quasi-variational inequality: find $\bar{x} \in D$ such that

$$\varphi(\bar{x}, y) + \Sigma(\bar{x}, y) - \Sigma(\bar{x}, \bar{x}) \geq 0 \quad \forall y \in D. \tag{2.1}$$

In the present paper, having in mind some applications, we focus on the so-called "obstacle problem" : given $K : D \rightrightarrows X$, find $\bar{y} \in K(\bar{y})$ such that

$$\varphi(\bar{x}, y) + \Phi(y) - \Phi(\bar{x}) \geq 0, \quad \forall\, y \in K(\bar{x}), \tag{2.2}$$

where $K : D \rightrightarrows X$, is a multivalued application from D to 2^X (this is the meaning of the notation "$D \rightrightarrows X$") and Φ is an extended real-valued function from D to $\mathbb{R} \cup \{+\infty\}$; here we have set $\Sigma(\bar{x}, y) = 1_{K(\bar{x})} + \Phi(y)$, where 1_C denotes the indicator function of the set C:

$$1_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{else.} \end{cases}$$

Problem (2.1) was considered earlier by Mosco and Joly [16] in regard to existence theory in the coercive setting, and recently studied by Bazán [13] again from the point of view of existence in the noncoercive framework. Problem (2.1) covers other problems more than those quoted in these previous works: it still therefore deserves a further treatment. We have to notice that the function Σ in (2.1) has been introduced to reflect the dependence with respect to the constraints on the solutions while the term $\Phi(y) - \Phi(\bar{x})$ can not be contained in φ since Φ may takes the infinity as a value. As confirmed by the existing literature, from the stability point of view, only few efforts have been dedicated to quasi-variational inequalities. Some qualitative results were established by Morgan and Lignola in [18] for the case $\Phi = 0$ and the obtained properties can be regarded as a closeness of the solution map, which is intimately related to *upper approximation* of solutions.

In [17] QVI solutions existence was considered via Tychonov well-posedness tool. The paper is henceforth devoted to the well-posedness properties of QVI. This leads to existence/uniqueness results but assumptions and techniques are quite different from ours. From another point of view, in [3] the authors focus on the (differential) set-valued operator (say T) defining the QVI. They investigate quasi-monotonicity properties that we do not consider in this paper. We assume (in as standard way) that the operator (A) is single-valued and monotone.

Note that results of [1] cannot be applied here since the framework is completely different. We deal with a general constraint set-valued operator K and look for existence results (via stability) in infinite dimensional spaces, that is not the case in [1] .

Throughout this paper V will be a reflexive Banach space whose topological dual, duality pairing and norm are denoted by V', $\langle ., . \rangle$ and $\|.\|$ respectively. The norm of V' will be denoted by $\|.\|_*$. The symbol $\rightarrow$ (resp. $\rightharpoonup$) will stand for the strong (resp. weak) convergence. The control function (that is fixed in a first step) u belongs to an Hilbert space U. Let us give

- operators $A : V \rightarrow V'$ and $B : U \rightarrow V'$
- a set-valued map $K : V \rightrightarrows V$ with nonempty closed convex values. Note that K may involve a "constant" part that may represent classical state constraints. For example, $K(y) = \tilde{K}(y) \cap C$ where $\tilde{K} : V \rightrightarrows V$ and C is a non empty, convex subset of V.
- a convex extended real-valued function $\Phi : V \times U \rightarrow \mathbb{R} \cup \{+\infty\}$, whose properties will be made precise in the sequel. Note that the convex C mentioned above can be the domain of Φ.

The corresponding quasi-variational inequality problem, QVI (A, u, Φ, K), is to find $\bar{y} \in K(\bar{y})$ such that

$$\langle A\bar{y} - Bu, y - \bar{y} \rangle + \Phi(y, u) - \Phi(\bar{y}, u) \geq 0, \quad \forall\, y \in K(\bar{y}), \tag{2.3}$$

Many problems arising in optimization, economic equilibrium [14], calculus of variations, free boundary problems [4], feasibility in optimal control as well as in mechanic [12] can be modelled by (2.3). We shall assume that

$$\forall x \in V \quad 0 \in K(x) \tag{2.4}$$

This assumption is consistent with the applications we present at the end of this paper. We recall now some variational analysis basic concepts, that we need in the sequel.

2.2 Hemicontinuity, semicontinuity and monotonicity

Let X and Y be two Hausdorff topological spaces and let $\Gamma : X \rightrightarrows Y$ be a set-valued map. Recall that the domain of Γ is $\mathrm{Dom}\,(\Gamma) = \{\mathrm{x} \in \mathrm{X} | \, \Gamma(\mathrm{x}) \neq \emptyset\}$. Its graph is the set

$$\mathrm{graph}\,(\Gamma) = \{(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y} | \ \mathrm{x} \in \mathrm{Dom}\,(\Gamma),\ \mathrm{y} \in \Gamma(\mathrm{x})\}.$$

- If graph (Γ) is closed (resp. convex), we say that Γ is *closed* (resp. convex).
- Γ is *upper (resp. lower) semicontinuous* at $\overline{x} \in X$ if for any open U in Y with $\Gamma(\overline{x}) \subset U$ (resp. $\Gamma(\overline{x}) \cap U \neq \emptyset$) the set $\{x \in X \mid \Gamma(x) \subset U\}$ (resp. $\{x \in X \mid \Gamma(x) \cap U \neq \emptyset\}$) is open in X.
- Γ is said to be *continuous* at $\overline{x}$ if it is both upper and lower semicontinuous at $\overline{x}$.

Let us mention that if Γ is upper semicontinuous, then it is closed. If in addition, the range of Γ is compact, then Γ is upper semicontinuous if and only if Γ is closed. For a discussion on this topic and related continuity properties we refer for example to [2,8] and references cited therein.
A single valued operator $A : X \to X'$ is said to be *hemicontinuous* [5] if, for all $u, v \in X$,

$$w - \lim_{\lambda \to 0} A(u + \lambda v) = Au.$$

An operator $A : X \to X'$ is said to be *monotone* if

$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad \forall u, v \in X \tag{2.5}$$

and M-strongly monotone if there is some $M > 0$ such that

$$\langle A(u) - A(v), u - v \rangle \geq M \|u - v\|^2 \quad \forall u, v \in X. \tag{2.6}$$

Most of existence schemes of solutions to quasi-variational inequalities use the recurrent tool : define a suitable set-valued map related to the data of the problem and look for its fixed points. There are many results of fixed points of set-valued maps e.g. the Kakutani-Ky Fan's result which is an extension of the topological Brouwer's fixed point: any self upper semicontinuous set-valued map with nonempty compact and convex values has a fixed point, (see [4,23] for example).

2.3 Mosco convergence

Let $(K_n)_n$ be a sequence of subsets of V. We recall basic definitions on set convergence:

Definition 1 Let $(K_n)_n$ be a sequence of nonempty closed convex subsets of V. We say that K_n converges to K (a closed convex subset of V) in the sense of Mosco if the following two assumptions are satisfied

(i) If $(v_n)_n$ weakly converges to v with $v_n \in K_n$ for n large enough, then the weak limit $v \in K$;
(ii) For any $v \in K$, there exists a sequence $(v_n)_n$ strongly converging to v such that $v_n \in K_n$ for every n large enough.

Remark 1 The above assumptions (i) and (ii) in Definition 1 can be summarized as

$$w - \limsup_n K_n \subseteq K \text{ and } s - \liminf_n K_n \supseteq K, \tag{2.7}$$

where $s-\liminf$ and $w-\limsup$ denotes the inferior strong limit and the superior weak limit respectively in the sense of Kuratowski-Painlevé. We note also that, since strong convergence implies weak convergence, (i) and (ii) are equivalent to

$$w - \limsup_n K_n \subseteq K \subseteq w - \liminf_n K_n \text{ and } s - \limsup_n K_n \subseteq K \subseteq s - \liminf_n K_n.$$

We can find some further characterizations of the Mosco-convergence by using projections, distance functions and the convergence of Yosida approximations in [11,20,22].

In the very case where the sequence K_n is related to a set-valued application we may formulate the Mosco-convergence as follows:

Definition 2 Let $K : V \rightrightarrows V$ a set-valued application. For any $v \in V$ and any sequence $(v_n)_{n\in\mathbb{N}}$ (weakly) converging to v, we say that the sequence of sets $K(v_n)$ Mosco-converges to $K(v)$ if and only if:

(i) For every sequence $y_n \in K(v_n)$ such that y_n weakly converges to y, then $y \in K(v)$.
(ii) For every $y \in K(v)$, there exists $y_n \in K(v_n)$ (for n large enough) such that y_n strongly converges to y.

In Sect. 5. we give an example of Mosco-convergence related to the control problem we study in the sequel.

3 Existence of solutions to QIV Problem (2.3)

In what follows the control function u is fixed and we set $f = Bu \in V'$ in the sequel. For the sake of simplicity, we do not indicate the dependence of Φ with respect to u and denote $\Phi(., u) := \Phi_u$ for the fixed value of the control parameter $u \in U$.
Let us introduce the map $\mathbb{S}_u : V \rightrightarrows V$ defined by

$$\mathbb{S}_u(x) = \{y \in V \mid \langle Ay - f, z - y\rangle + \Phi_u(z) - \Phi_u(y) \geq 0, \ \forall z \in K(x)\}, \tag{3.1}$$

or equivalently

$$\mathbb{S}_u(x) = \bigcap_{z\in K(x)} F_x(z),$$

where $F_x(z) = \{y \in V \mid \langle Ay - f, z - y\rangle + \Phi_u(z) - \Phi_u(y) \geq 0\}$.
Clearly, the solutions of problem (2.3) are fixed points of the map $\mathbb{S}_u$. Therefore, the scheme of existence of solutions to this problem is based on the two following essential steps:

– $\mathbb{S}_u$ is nonempty-valued i.e., $\bigcap_{z\in K(x)} F_x(z) \neq \emptyset$ for every x;
– $\mathbb{S}_u$ admits at least a fixed point.

When it is nonempty, the set of fixed points of $\mathbb{S}_u$ will be denoted by $\mathbf{FP}(\mathbb{S}_u)$. The operator A, a closed convex subset D of V and $f \in V'$ being given, we call $\mathcal{S}_u(D)$ the solutions set to the following variational inequality VI(D): find $y \in D$ such that

$$\langle Ay - f, z - y\rangle + \Phi_u(z) - \Phi_u(y) \geq 0, \ \forall z \in D. \tag{3.2}$$

We look for the fixed points of $\mathbb{S}_u$ where $\mathbb{S}_u(x) = \mathcal{S}_u(K(x))$. If monotonicity and convexity assumptions occur, a classical tool is to consider the Minty's variational inequality: find $y \in D$ such that

$$\langle Az - f, z - y\rangle + \Phi_u(z) - \Phi_u(y) \geq 0, \ \forall z \in D. \tag{3.3}$$

Let us call $\mathcal{S}_u^M(D)$ the solutions set of (3.3).

Lemma 1 *If Φ_u is proper, convex and lower-semicontinuous, then for every $x \in V$, $\mathcal{S}_u^M(K(x))$ is closed and convex, possibly empty. In addition, if A is monotone, hemicontinuous then*

(i) $\mathcal{S}_u(K(x)) = \mathcal{S}_u^M(K(x))$.
(ii) *$\mathbb{S}_u$ is closed and convex valued on its domain.*

Proof The first point is a classical result (see [4,8] for example). Point (ii) is a direct consequence of point (i). □

We turn now our attention to existence results for problem (2.3) when the operator A is strongly monotone and hemicontinuous and $u \in U$ is fixed.

Theorem 1 *Let be $u \in U$ Assume the following holds:*

(i) *$A : V \to V'$ is hemicontinuous and M-strongly monotone;*
(ii) *A is bounded and*

$$\forall y_n \to y, \ \forall z_n \rightharpoonup z \qquad \langle A(y), z - y\rangle \leq \liminf_n \langle A(y_n), z_n - y_n\rangle.$$

(iii) *For all $(x_n)_n$ in V such that $x_n \rightharpoonup x$, then $K(x_n)$ Mosco-converges to $K(x)$;*
(iv) *We assume that $\Phi_u : V \to \mathbb{R}$ is convex and continuous and either*

  (a) *Φ_u is L-Lipschitz continuous with $L > 0$*
  or
  (b) *Φ_u satisfies*

$$\min_{v \in V} \Phi_u(v) = \Phi_u(0).$$

Then problem (2.3) admits at least one solution.

Proof

- We first prove that $\mathbb{S}_u$ is (graph) weakly- closed, that is:

$$\text{if } (x_n, y_n) \rightharpoonup (x, y) \text{ with } y_n \in \mathbb{S}_u(x_n) \text{ then } y \in \mathbb{S}_u(x). \tag{3.4}$$

  Let $(x_n)_{n \in \mathbb{N}}$ be a sequence weakly convergent to $x \in V$. Thanks to assumptions (i) and (iv) we know that

$$\forall n \qquad \mathcal{S}_u(K(x_n)) \neq \emptyset,$$

  (one can refer to [5] for example). The strong monotonicity of A ensures the uniqueness of the solution to VI $(A, f, \Phi_u, K(x_n))$ for every n so that

$$\mathbb{S}_u(x_n) = \{y_n\}\ .$$

We assume that $y_n \rightharpoonup y$; we have to prove that $y \in \mathbb{S}_u(x)$. Let z be an arbitrary point of $K(x)$. From (iii) there exists $z_n \in K(x_n)$ such that $z_n \to z$ strongly in V. This implies that $\lim_{n\to+\infty} \Phi_u(z_n) = \Phi_u(z)$. As $y_n \in \mathcal{S}_u(K(x_n))$ we get

$$\langle Ay_n - f, z_n - y_n\rangle + \Phi_u(z_n) - \Phi_u(y_n) \geq 0.$$

In addition $y_n \in \mathcal{S}_u^M(K(x_n))$ (cf. Lemma 1) so that

$$\langle Az_n - f, y_n - z_n\rangle + \Phi_u(y_n) - \Phi_u(z_n) \leq 0.$$

Moreover Φ_u is continuous and convex, so it is weakly lower semi-continuous so that $\Phi_u(y) \leq \liminf_n \Phi_u(y_n)$. Using (ii) this yields that

$$\langle Az - f, y - z\rangle + \Phi_u(y) - \Phi_u(z) \leq \liminf_n \langle Az_n - f, y_n - z_n\rangle + \liminf_n (\Phi_u(y_n) - \lim_n \Phi_u(z_n)) \leq 0.$$

So

$$\langle Az - f, z - y\rangle + \Phi_u(z) - \Phi_u(y) \geq 0, \quad \forall z \in K(x).$$

This means that $y \in \mathcal{S}_u^M(K(x)) = \mathcal{S}_u(K(x)) = \mathbb{S}_u(x)$.

- We prove now that $\mathbb{S}_u$ has at least a fixed point. We have shown at the beginning of the proof that

$$\forall x \in V \qquad \mathcal{S}_u(K(x)) = \mathbb{S}_u(x) = \{y_x\}.$$

Therefore $\mathbb{S}_u$ is *single-valued*. Moreover, $0 \in K(x)$ (assumption 2.4) implies that

$$\langle Ay_x - f, -y_x\rangle + \Phi_u(0) - \Phi_u(y_x) \geq 0.$$

Therefore, with the strong monotonicity of A, we obtain

$$M\|y_x\|^2 \leq \langle Ay_x - A(0), y_x\rangle \leq \Phi_u(0) - \Phi_u(y_x) + \langle f - A(0), y_x\rangle. \tag{3.5}$$

If assumption (iv)-(a) is satisfied, then using (3.5), we get

$$M\|y_x\|^2 \leq L\|y_x\| + \langle f - A(0), y_x\rangle.$$

Thus, $\|y_x\| \leq c\|f\|_*$ with $c = c_{f,L,M} = \dfrac{\|f - A(0)\|_* + L}{M}$ (c is independent of x).
If assumption (iv)-(b) is satisfied, then using (3.5), we have

$$M\|y_x\|^2 \leq \langle f - A(0), y_x\rangle.$$

Thus, $\|y_x\| \leq c\|f\|_*$ with $c = c_{f,M} = \dfrac{\|f - A(0)\|_*}{M}$ (c is independent of x).
Consider the convex weakly compact $C_0 := \overline{B}(0, c)$ of V : we observe that $\mathbb{S}_u(C_0) \subset C_0$. As $\mathbb{S}_u$ is single valued and weakly closed we may use Schauder-Tychonoff theorem (see e.g. [15] page 147 Theorem 1.10) to ensure the existence of a fixed point of $\mathbb{S}$ in C_0. □

Remark 2 The above result is a general existence result. In particular the assumption (iii) is a strong assumption which involves compactness of K. Many other existence results can be found in the literature without such hypothesis but only for some specific situations; as often the underlying compactness assumption is replaced by monotonicity assumptions. We refer to [6] for more details.

4 Stability results for problem (2.3) with respect to f perturbations

Although the generality is of great interest, we consider now the classical functional framework for PDE's control.

Let V, H be Hilbert spaces such that $V \subset H$ with continuous, compact and dense embedding. V' denotes the dual of V. The control space U is an Hilbert space as well and U_{ad} is the set of admissible control functions: it a non empty, convex closed subset of U. The quasi-variational problems: QVI(A, u, Φ, K) turns to be:

$$\text{find } y_u \in K(y_u)\colon\ \langle Ay_u - Bu, y - y_u\rangle + \Phi(y, u) - \Phi(y_u, u) \geq 0,\ \forall y \in K(y_u). \quad (4.1)$$

In the previous section, we proved that for every $u \in U$ the set of solutions to QVI(A, u, Φ, K) is non empty (under assumptions of Theorem 1). So we may define the solution map of problem (4.1) that we denote by

$$\mathcal{T} : u \mapsto \mathcal{T}(u)$$

which is set-valued. We now establish the weak sequential closeness of $\mathcal{T}$.

Theorem 2 *Let be $u \in U$ and $(u_n)_{n\in\mathbb{N}} \in U$ a sequence weakly convergent to u in U. Assume the following:*

- (i) *The set-valued map K is closed and convex valued, i.e. $K(w)$ is closed and convex for all $w \in V$;*
- (ii) *$A : V \to V'$ is hemicontinuous, strongly monotone and bounded;*
- (iii) *for every sequence $(y_n)_{n\in\mathbb{N}}$ strongly convergent to y in V and $(z_n)_{n\in\mathbb{N}}$ weakly convergent to z in V,*

$$\langle A(y), z - y\rangle \leq \liminf_n \langle A(y_n), z_n - y_n\rangle.$$

- (iv) *$B : U \to V'$ is a compact operator;*
- (v) *For every x_n such that x_n weakly converges to x in V, then $K(x_n)$ Mosco-converges to $K(x)$.*
- (vi) *$\Phi : V \times U \to \mathbb{R}$, is continuous and convex with respect to $y \in V$ and either,*
  - *Φ is L-Lipschitz continuous with respect to $y \in V$ uniformly with respect to u i.e.*

$$\forall u \in U \qquad |\Phi(y, u) - \Phi(z, u)| \leq L\,\|y - z\|, \quad (4.2)$$

  where $L > 0$ is independent on u,
  or

  -

$$\forall u \in U \qquad \min_{y\in V} \Phi(y, u) = \Phi(0, u). \quad (4.3)$$

  Moreover it must satisfies

  - (a) *For every $x_n \in \mathcal{T}(u_n)$ such that: $x_n \rightharpoonup x$ (weakly in V)*

$$\Phi(x, u) \leq \liminf_n \Phi(x_n, u_n).$$

  - (b) *For every $x_n \in \mathcal{T}(u_n)$ such that: $x_n \to x$ (strongly in V), there are subsequences (x_{n_k}) and (u_{n_k}) of (x_n) and (u_n) respectively such that*

$$\limsup_k \Phi(x_{n_k}, u_{n_k}) \leq \Phi(x, u).$$

Then

1. *There exists a constant* κ_u *depending on* u *such that*

$$\bigcup_{n\in\mathbb{N}} \mathcal{T}(u_n) \subset \mathcal{B}(0, \kappa_u); \tag{4.4}$$

   where $\mathcal{B}(0, \kappa_u)$ *is the* V*-ball of radius* κ_u*.*
2. *For every* $y_n \in \mathcal{T}(u_n)$*, the sequence* y_n *weakly converges (up to a subsequence) to some* $y \in \mathcal{T}(u)$*.*

Remark 3 Note that

– assumptions (i)–(v) are global assumptions on operators A, K and B while (vi) is a local one depending on u as a limit point of a sequence $(u_n)_{n\in\mathbb{N}}$.
– assumptions of Theorem 1 have been involved in those of Theorem 2 so that

$$\forall u \in U \qquad \mathcal{T}(u) \neq \emptyset.$$

Proof We first prove (4.4).

With (2.4) and the convexity of $K(y_n)$ we claim that for any $t \in [0, 1[$ then $ty_n \in K(y_n)$. As $y_n \in \mathcal{T}(u_n)$ we get

$$\forall\, y \in K(y_n) \qquad \langle Ay_n - Bu_n, y - y_n\rangle + \Phi(y, u_n) - \Phi(y_n, u_n) \geq 0, \tag{4.5}$$

so that (with $y = ty_n$)

$$\forall n \in \mathbb{N}, \forall t \in [0, 1] \qquad (t-1)\langle Ay_n - Bu_n, y_n\rangle + \Phi(ty_n, u_n) - \Phi(y_n, u_n) \geq 0.$$

Now we use the first part of assumption (vi). Assuming (4.2), we obtain

$$|\Phi(ty_n, u_n) - \Phi(y_n, u_n)| \leq L(1-t)\|y_n\|;$$

so

$$\forall n \in \mathbb{N}, \forall t \in [0, 1] \qquad (t-1)\langle Ay_n - Bu_n, y_n\rangle + L(1-t)\|y_n\| \geq 0,$$

and

$$\forall n \in \mathbb{N}, \qquad \langle Ay_n - A(0), y_n\rangle + \langle A(0) - Bu_n, y_n\rangle - L\|y_n\| \leq 0.$$

We conclude with the strong monotonicity of A that

$$M\|y_n\|^2 \leq (\|A(0)\|_* + L + \|Bu_n\|)\|y_n\|. \tag{4.6}$$

Alternatively, let us assume (4.2b); with (4.5) an $y = 0$ we get

$$\begin{aligned}
&\forall n \in \mathbb{N}, \qquad \langle Ay_n - Bu_n, -y_n\rangle \geq \Phi(y_n, u_n) - \Phi(0, u_n) \geq 0,\\
&\forall n \in \mathbb{N}, \qquad \langle Ay_n - Bu_n, y_n\rangle \leq 0,\\
&\forall n \in \mathbb{N}, \qquad \langle Ay_n - A(0), y_n\rangle + \langle A(0) - Bu_n, y_n\rangle \leq 0,
\end{aligned}$$

Once again the strong monotonicity of A yields

$$M\|y_n\|^2 \leq (\|A(0)\|_* + \|Bu_n\|)\|y_n\|. \tag{4.7}$$

As the sequence (u_n) is bounded and B is compact we get (with 4.6, respectively 4.7)

$$\|y_n\| \leq \kappa(u),$$

where κ is a constant depending on u.

We now show point 2: let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}(u_n)$ such that $y_n \rightharpoonup y$ in V. We have to prove that $y \in \mathcal{T}(u))$.

First, observe that $y_n \in K(y_n)$: so with (v), y is an element of $K(y)$. Now, consider an arbitrary point z in $K(y)$. Since $K(y)$ is convex, for all $t \in]0,1]$, $z_t := tz+(1-t)y \in K(y)$. We claim that

$$\langle A(z_t) - Bu, y - z_t\rangle + \Phi(y,u) - \Phi(z_t,u) \leq 0. \tag{4.8}$$

Indeed, according to (v) one can find a sequence $(z_n)_n$ (strongly) converging to z_t such that $z_n \in K(y_n)$, $\forall n \geq 1$. Since $y_n \in \mathcal{T}(u_n)$, for every $n \geq 1$, we have

$$\langle A(y_n) - Bu_n, z_n - y_n\rangle + \Phi(z_n,u_n) - \Phi(y_n,u_n) \geq 0.$$

Using the monotonicity of A it follows that

$$\langle A(z_n) - Bu_n, z_n - y_n\rangle + \Phi(z_n,u_n) - \Phi(y_n,u_n) \geq 0.$$

The operator $B : U \to V'$ is compact, so $(Bu_n)_n$ strongly converging to Bu in V'(up to a subsequence). Using hypothesis (iii) and (vi), up to subsequences if necessary, we obtain

$$\begin{aligned}\langle A(z_t) - Bu, y - z_t\rangle + \Phi(y,u) - \Phi(z_t,u) &\leq \liminf_n \langle A(z_n) - Bu_n, y_n - z_n\rangle \\ &\quad + \liminf_n \Phi(y_n,u_n) - \limsup_n \Phi(z_n,u_n) \leq 0.\end{aligned}$$

So, we have proved relation (4.8). We conclude that,

$$\langle A(z_t) - Bu, z_t - y\rangle + \Phi(z_t,u) - \Phi(y,u) \geq 0.$$

Using the convexity of Φ with respect to the first variable, we get

$$\langle A(z_t) - Bu, t(z - y)\rangle + t[\Phi(z,u) - \Phi(y,u)] \geq 0;$$

this gives

$$\forall t \in]0,1] \qquad \langle A(z_t) - Bu, z - y\rangle + \Phi(z,u) - \Phi(y,u) \geq 0, \tag{4.9}$$

We conclude with the hemicontinuity of A that

$$\langle A(y) - Bu, z - y\rangle + \Phi(z,u) - \Phi(y,u) \geq 0 \tag{4.10}$$

As the inequality (4.10) holds for any $z \in K(y)$, we conclude that $y \in \mathcal{T}(u)$; this achieves the proof. □

Example 1 Let us give a simple case where assumption (v) of Theorem 2 is satisfied . We define the set-valued mapping $K : V \rightrightarrows V$ by

$$K(v) = K_o + m(v) \tag{4.11}$$

where K_o is a fixed closed convex and nonempty subset of V and $m : V \to V$ is a compact map (or equivalently weakly - strongly continuous).

Proposition 1 *For any sequence (v_n) weakly convergent to v in V, then $K(v_n)$ Mosco-converges to $K(v)$, where K is given by* (4.11).

Proof Let (v_n) be a sequence of V such that $v_n \rightharpoonup v$, we have to prove that $K(v_n)$ Mosco-converges to $K(v)$. Let $w_n \in K(v_n)$ such that $w_n \rightharpoonup w$ in V. Since the operator $v \mapsto m(v)$ is compact $m(v_n) \rightharpoonup m(v)$. Since K_o is weakly closed then, $w - m(v) \in K_o$. Hence, $w \in K(v)$.

Let $w \in K(v)$, then $\exists k \in K_o$ such that: $w = k + m(v)$. We set $w_n = k + m(v_n) \in K(v_n)$. It is clear that $w_n \to k + m(v) = w$. Therefore, $K(v_n)$ Mosco-converges to $K(v)$. □

We will give in Sect. 5, some examples of such set-valued mapping K.

5 The optimal control problem

5.1 Existence result

Now we turn back to problem $(\mathcal{P})$ mentioned in Sect. 1. We suppose that assumptions of Theorem 1 are satisfied so that the set-valued operator $\mathcal{T}$ (defined in the previous sections) is well defined on U.

Consider a cost functional $J : V \times U \to \mathbb{R} \cup \{+\infty\}$, we set

$$(\mathcal{P}) \qquad \min\{J(y, u),\ y \in \mathcal{T}(u),\ u \in U_{ad}, \subset U\},$$

where U_{ad} is a non-empty, convex and closed subset of the (Hilbert) space U.

Theorem 3 *Assume J is convex and lower-semicontinuous and either U_{ad} is bounded or J is coercive with respect to u. Assume assumptions of Theorem 2 are satisfied for every cluster point of minimizing sequences of problem $(\mathcal{P})$. Then problem $(\mathcal{P})$ has at least one optimal solution.*

Proof The proof is straightforward. Let $(u_n)_{n\in\mathbb{N}} \in U_{ad}$ be a minimizing sequence. The boundedness of U_{ad} or the coercivity of J implies that u_n is bounded in U. Let u be a weak-cluster point of $(u_n)_{n\in\mathbb{N}}$ and denote the corresponding subsequence similarly. Assumptions of Theorem 2 are satisfied so that for every $y_n \in \mathcal{T}(u_n)$, y_n is a bounded sequence and weakly converges (up to a subsequence) to $y \in V$. Moreover $y \in \mathcal{T}(u)$.

We end the proof with the lower semi-continuity of J □

In the sequel, using [19] formalism we choose J as follows:

– $\mathcal{N} : \mathcal{U} \to U$ is a linear, symmetric continuous and coercive operator i.e.

$$\exists \kappa > 0 :\ \langle \mathcal{N} v, v\rangle \geq \kappa \|v\|^2, \quad \forall v \in U$$

– $\mathcal{H}$ is a Hilbert space of observations and $\mathcal{C} \in \mathcal{L}(V, \mathcal{H})$ be a given continuous operator.
– The desired state is $z_d \in \mathcal{H}$

We associate the following cost functional $J : V \times U \to \mathbb{R}$ defined by

$$J(y, u) = \langle \mathcal{N} u, u\rangle_U + \|Cy - z_d\|^2_{\mathcal{H}}. \tag{5.1}$$

5.2 Some relevant examples

In this subsection, we will give some simple examples to support our theoretical results.

Example 2 Implicit Signorini problem

Let $\Omega \subset \mathbb{R}^n$ an open bounded connected set with a regular boundary $\partial\Omega = \Gamma$. Let us consider the following implicit Signorini problem

$$\begin{cases} \text{Find } y \in K(y) \text{ such that} \\ a(y, z - y) \geq \langle u, z - y\rangle, \quad \forall z \in K(y) \end{cases} \tag{5.2}$$

with the following data

$$V = \{y \in H^1(\Omega) \mid \Delta y \in L^2(\Omega)\},$$

for each $y \in V$, we associate the closed convex non empty set of $H^1(\Omega)$ defined by

$$K(y) = \left\{ z \in H^1(\Omega) \mid \; z_{|\Gamma} \geq h - \int_\Gamma \varphi \frac{\partial y}{\partial n} d\sigma \text{ a.e. on } \Gamma \right\},$$

with $h,\ \varphi \in H^{\frac{1}{2}}(\Gamma)$ and $h \geq 0$ on Γ.

$$a(y, z) = \int_\Omega \left(\nabla y \cdot \nabla z + y\, z\right) dx$$

$$\langle u, z\rangle = \int_\Omega u\, z\, dx, \quad u \in L^2(\Omega).$$

It is known that the solution y of (5.2) (Implicit Signorini problem) is characterized by

$$\begin{cases} -\Delta y + y = u \text{ a.e. in } \Omega \\ y \geq \left(h - \int_\Gamma \varphi \frac{\partial y}{\partial n} d\sigma\right), \quad \frac{\partial y}{\partial n} \geq 0, \quad \left[y - \left(h - \int_\Gamma \varphi \frac{\partial y}{\partial n} d\sigma\right)\right] \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma. \end{cases}$$

We have to verify assumptions of Theorem 2:

- Assumption (i) is satisfied (we refer to [16] p.130).
- It is clear that, (ii) and (iii) are ensured (with $A = -\Delta + Id$).
- $U = H = L^2(\Omega)$ and $B : H \to V'$ is the canonical (compact) embedding, so we get (iv).
- Let us show that (v) is satisfied. We first note that K is defined by (4.11) with

  $$K_o = \{z \in H^1(\Omega) \mid z_{|\Gamma} \geq h \text{ a.e. on } \Gamma\}$$

  and $m : V \to V$ is given by $m(y) = -\int_\Gamma \varphi \frac{\partial y}{\partial n} d\sigma$, (here real numbers are identified to constant functions). We have to prove that m is compact. Let y_k be a sequence of V weakly convergent to y. The normal derivative trace operator is linear, continuous (and thus weakly continuous) from V to $H^{-\frac{1}{2}}(\Gamma)$ [21] so that $m(y_k) \to m(y)$ in $\mathbb{R}$. As the constant functions space (identified to $\mathbb{R}$) is (compactly) embedded in V this gives the strong convergence of $m(y_k)$ to $m(y)$ in V.
- $\Phi = 0$ so that (vi) is fulfilled.

We may summarize:

Theorem 4 *Assume $\alpha > 0$. Then the optimal control problem*

$$\begin{cases} \min \ \|y - z_d\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\Omega)} \\ -\Delta y + y = u \text{ a.e. in } \Omega \\ y \geq \left(h - \int\limits_\Gamma \varphi \dfrac{\partial y}{\partial n} d\sigma\right), \quad \frac{\partial y}{\partial n} \geq 0, \quad \left[y - \left(h - \int\limits_\Gamma \varphi \frac{\partial y}{\partial n} d\sigma\right)\right] \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma. \\ u \in U_{ad} \end{cases}$$

has (at least) one optimal solution.

Example 3

Let $\Omega \subset \mathbb{R}^n$ an open bounded connected set with a regular boundary $\partial\Omega = \Gamma$. We consider (5.2) with

$$V = H^1(\Omega), \ a(y, z) = \int\limits_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial y}{\partial x_i}(x) \frac{\partial z}{\partial x_j}(x) + b(x) y(x) z(x)\right] dx,$$

where the functions $x \mapsto a_{ij}(x)$ and $x \mapsto b(x)$ satisfy the following classical assumptions:

$$a_{ij} \in L^\infty(\Omega), \ 1 \leq i, j \leq n, \quad b \in L^\infty(\Omega), \ b \geq 0 \text{ a.e. on } \Omega$$

and

$$\exists \beta > 0, \ \forall \xi_i, \ 1 \leq i \leq n, \ \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \beta \sum_{i=1}^n \xi_i^2 \text{ a.e. on } \Omega.$$

$$K(y) = \left\{ z \in H^1(\Omega) \mid \int\limits_\Omega (z - y)(x)\, dx \geq 0 \right\}.$$

It is easy to check all assumptions of Theorem 2 except (v). Assume that y_k is a sequence of $H^1(\Omega)$ weakly convergent to y. Let us prove that $K(y_n)$ Mosco-converges to $K(y)$:

(i) Assume that $z_n \in K(y_n)$ weakly converges to z in $H^1(\Omega)$. So $(z_n - y_n, 1)_{L^2(\Omega)} \geq 0$ and obviously converges to $(z - y, 1)_{L^2(\Omega)}$. Therefore $\int\limits_\Omega (z - y)(x)\, dx \geq 0$ that is $z \in K(y)$.

(ii) Let $z \in K(y)$. We set $z_n = z + \int\limits_\Omega (y_n - y)(x) dx$. Obviously z_n strongly converges to z in $H^1(\Omega)$. Moreover

$$\begin{aligned} \int\limits_\Omega (z_n - y_n)(x)\, dx &= (z_n - y_n, 1)_{L^2(\Omega)} \\ &= (z + (y_n - y, 1)_{L^2(\Omega)} - y_n, 1)_{L^2(\Omega)} \\ &= (z - y_n, 1)_{L^2(\Omega)} + (y_n - y, 1)_{L^2(\Omega)} \\ &= (z - y, 1)_{L^2(\Omega)} \geq 0, \end{aligned}$$

that is $z_n \in K(y_n)$.
Now it is easy to conclude as previously.

Example 4 Implicit Obstacle problem with friction

Let $\Omega \subset \mathbb{R}^2$ be the section of a tube with infinity length $\Omega\times]-\infty, +\infty[$ and a thin membrane spanned in the tube held by a thin elastic wire. We assume that Ω is an open bounded connected set with a regular boundary $\partial\Omega$. The membrane deforms under the action of a surface density force f acting in the axe-direction and is required to stay on or above an obstacle $m(u)$ where $m : H^1(\Omega) \to H^1(\Omega)$ is a compact operator. The forces f are balanced by the force that the obstacle exerts on when the membrane is in contact with it. The displacement of the membrane is governed by the quasi-variational inequality (5.2) where

$$V = H^1(\Omega),\ K(y) = \{z \in V \mid z \geq m(y)\},\ a(y, z) = \int_\Omega \Big(\nabla y \cdot \nabla z + y\, z\Big) dx$$

$$\langle f, y\rangle = \int_\Omega f y\, dx \quad \text{and} \quad \Phi(y) = \int_\Gamma g\,|y|\, d\sigma,$$

with $g \in L^\infty(\Gamma)$, $g \geq 0$ on Γ a given friction bound.
Assumptions of Theorem 2 are satisfied. The function Φ is convex and Lipschitz continuous and it is easy to see that

$$K(y) = K_o + m(y),$$

as in (4.11) with

$$K_o = \{z \in H^1(\Omega) \mid z \geq 0 \text{ a. e. in } \Omega\},$$

So we conclude that Theorem 2 applies.

6 Conclusion

We have given an abstract framework that allows to consider optimal control problems governed by Quasi-Variational Inequalities. The strong monotonicity of the operator A and the Mosco-convergence of the set-valued application K involved in the QVI are key assumptions in the proofs presented in this paper. Since in some applications the operator A is only semi-coercive and since the underlying compactness assumption (involved in the Mosco-convergence of K) does not allow to deal with “usual ” constraints as pointwise constraints for example. It would be very interesting to obtain similar results with only a monotonicity assumption insteact of a compactness one. This is out of the scope of the present manuscript and will be the subject of another work.

Acknowledgements The research of S. Adly has been supported by the “Fondation EADS” and the ANR project “Guidage”.

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